The Excel Math Competition

April 2025

1. This is a 45 minute individual exam.
2. No collaboration is allowed.
3. The first 10 questions are worth [5] points each and are multiple choice.
4. The last 5 questions are worth [10] points each and are short response.
5. Each of the final 5 questions have answers which are positive integers between 000 and 999, inclusive.
6. The questions are arranged in roughly ascending difficulty.
7. If you believe a question is seriously flawed, or have an answer which is not one of the listed answers, there will be a 10-minute dispute period after the test, after which no disputes will be accepted.
8. In the event of a dispute, **leave the question blank** and let your proctor know after the testing time ends.
9. All disputes will be considered on an individual-by-individual basis, so no student will receive credit if they did not submit a dispute, except for in the case of a question being thrown out.

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1 [5]

Welcome to ExcelAcademe's April competition! In the spirit of the month, consider the adage "April showers bring May flowers." If the odds of an April shower occurring on any one of the 30 independent days in April are $7:5$, and $k \cdot n$ May flowers will bloom on the n'th day in May (which has 31 days), where k is the total number of April showers which occurred between April 1st and April $n-1$'th, then the expected value of the number of May flowers that will bloom in all of May is a simplified fraction $\frac{a}{b}$. Find the sum of the distinct prime factors of $a \cdot b$

a) 12 b) 15 c) 29 d) 48 e) 50

2 [5]

Triangle ABC has $AB = 4, BC = 4, AC = 2$ and incenter I. Three distinct circles are drawn with diameters having endpoints I and one of A, B, C. The centers of these circles are connected to form a triangle of area $\frac{\sqrt{a}}{b}$ in most simplified form. Find $a \cdot b$

a) 40 b) 50 c) 60 d) 75 e) 100

3 [5]

Myopia (nearsightedness) in the human eye is quantified using the unit of *Diopters*, measured in meters^{-1}, and calculated as the reciprocal of the focal length of the eye. Simple myopia is diagnosed at $\leqslant 4$ Diopters. If the human eye is modeled as an ellipsoid (3-D analog to an ellipse) which is radially symmetric about the axis of incoming light rays, and with semiminor axes of length 1, then the minimum volume of the eyeball needed to be diagnosed with simple myopia is given by $\frac{\sqrt{a}\cdot\pi}{b}$ with a not divisible by the square of any integer. Find $a \cdot b$

a) 54 b) 58 c) 62 d) 64 e) 68

4 [5]

Find the sum of the real solutions to $(x^2 - 3x + 3)^{x^2+2x-15} = 1$

a) -2 b) -1 c) 0 d) 1 e) 2

5 [5]

It's Mu Alpha Theta nationals, and President of FAMAT Bobby Hail is feeling the uncontrollable urge to wear his signature hat to hide his embarrassing hairline. In fact, President Hail's hairline can be modeled by a piecewise function on the interval $[-10, 10]$:

$$\text{Chopped\_Hairline}(x) := \begin{cases} \frac{3}{2}x + 15, & x \in [-10, -6) \\ \frac{1}{6}x^2, & x \in [-6, 6) \\ -\frac{3}{2}x, & x \in [6, 10] \end{cases}$$

Correspondihngly, the region occupied by President Hail's hair is all y such that $20 \geqslant y \geqslant \text{Chopped\_Hairline}(x)$. Hail places his circular hat on his head, only to realize that there is always a placement of the hat on his head (the plane) which covers none of his devious hairline (all y not satisfying the above relation)! The the maximum area of President Hail's circular hat can be written as $\pi \cdot \frac{a}{b}$ in most simplified form. Find $a - 2b$

a) 601 b) 991 c) 1236 d) 1237 e) 4051

6 [5]

Mr. Sillygirg is tired of his traditional combinatorics questions asking how many permutations of a word's letters there are, so he decides to step it up a notch. He comes up with the following question for one of the FAMAT tests he may or may not be writing:

The letters in the word JOEYIAMDENYINGYOURDISPUTE are arranged in a 5 by 5 square. How many arrangements are possible such that some row has the uninterrupted string of letters JOEY somewhere in it?

In the prime factorization of the answer to Mr. Sillygirg's question, the exponent of 2 is a, the exponent of 3 is b, and the exponent of 5 is c. Find $a + b + c$

a) 22 b) 24 c) 25 d) 28 e) 31

7 [5]

An equiangular convex octagon has alternating side lengths of a and b, with $a \geqslant b$. The maximum area of a triangle with vertices chosen from the vertices of the octagon is a function $f(a, b)$. The value of $\frac{f(50,50)}{f(13,12)}$ can be written in simplified form as $\frac{p_1^{e_1} p_2^{e_2} (p_3^{e_3} - \sqrt{2})}{p_4 p_5 p_6}$ for primes p_i. Find $p_1 p_2 - p_3$

a) -5 b) -3 c) -1 d) 1 e) 5

8 [5]

Let $f(n, l)$ be the number of integers $k > 1$ with $k \leqslant l$ such that the positive integer l when written in base k has n digits. Compute $\sum_{i=2}^{11} f(i, 2025)$

a) 2022 b) 2023 c) 2024 d) 2025 e) 2026

9 [5]

Dou Dizhu is a common card game originating in China. There are 3 players, each with 1/3 of the total deck (the deck includes a red and black joker for a total of 54 cards). Two of the players- the farmers- try to beat the third- the landlord. One of the most valuable card combinations is four-of-a-kind, and the most valuable combination is two jokers. The cards are dealt out from a shuffled deck. The probability that the landlord has the pair of jokers in any given hand is $\frac{a}{b}$ in simplest form. Find $a + b$

a) 68 b) 176 c) 252 d) 1096 e) 1225

10 [5]

Let $M = \sum_{i=1}^{100}\sum_{j=i}^{101} ij$. Find the remainder when 2025^M is divided by 7

a) 1 b) 2 c) 3 d) 5 e) 6

11 [10]

There exists a smallest positive integer n such that $n!$ is divisible by 2025. Find $\binom{n}{\frac{n}{2}}$

12 [10]

There exists a set of cubic polynomials $f(x)$ with leading coefficient 1 satisfying $f(0) = 0$ and $f(1+\sqrt{3}) = f(1-\sqrt{3}) = 2p$ where p is a prime integer. Define $g(p)$ as $f(1)$ for some p. The maximum value of $g(p)$ across all p is M. Find M^2

13 [10]

We define the *minimum bounding box* of a solid as the cube of smallest volume which can entirely contain the solid. A right circular cone has base radius 1 and height $6 - \frac{\sqrt{3}}{2}$. What is the greatest integer less than or equal to the volume of its minimum bounding box?

14 [10]

Define the following functions:

$$f(x) = x^4 - 3x^3 - 5x^2 + 2x + 2$$

$$g(x) = x^3 + ax^2 + bx + c$$

Where a, b are real numbers. If p, q, r, s are the roots of $f(x)$, and t, u, v are the roots of $g(x)$, then the maximum value of $pt + qu + rv + s$ can be written as a function of a, b, and c called $h(a, b, c)$. Find the remainder when $(h(20, 25, 69))^2$ is divided by 1000

Hint: The Cauchy-Schwarz inequality states for two vectors $u = \langle u_1, u_2, ...\rangle$ and $v = \langle v_1, v_2, ...\rangle$, the dot product $u \cdot v = \langle u_1v_1, u_2v_2, ...\rangle$ is less than or equal to the product of the norms $\|u\|\|v\| = \sqrt{u_1^2 + u_2^2 + \ ...}\sqrt{v_1^2 + v_2^2 + \ ...}$

15 [10]

In the plane, David, Ishaan, and Hank are positioned at the points $(0, 0), (4, 0), (0, 4)$, respectively. Each of these fine gentlemen, every second, move one unit in a random direction along the coordinate axis they are situated upon. David can move along either the x or y axis, but not diagonally. They will each stop to eat some food and recuperate once they hit the origin (David stops when he's hit the origin after a positive number of seconds). The probability that they all stop within 6 seconds is $\frac{a}{b}$ in most simplified form. Find a